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THE ECCENTRIC LAMÉ PROBLEM: AN EQUIVALENT PERTURBED DOMAIN SOLUTION

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Abstract—Hollow elastic cylinders containing moderately eccentric holes subjected to internal pressures are analyzed by means of a perturbation method. Using the eccentricity as the perturbation parameter, results are obtained by means of a sequence of directly derivable solutions for an equivalent perturbed concentric domain subjected to appropriate boundary tractions. The latter may be considered as corrective tractions which are required to satisfy the boundary conditions of the actual domain. Expressions for the stress field are established. Numerical results for the effect of eccentricity are presented for the stresses at the critical section and stress amplification factors are obtained. From a comparison of the results for the maximum stress components with an exact solution previously derived, it is seen that the method of Equivalent Perturbed Domains leads to accurate solutions for small to moderate eccentricities.

1. INTRODUCTION

Among the component parts of nuclear reactors, there exist thick-wall pipes containing fluids under pressure and high temperatures. It is known that the pipes, which initially have thick walls, are subject to considerable erosion. Due to an uneven erosion, these walls become much thinner over the years and the internal boundary, initially concentric, can become eccentric. High stresses, induced in the weakened walls, thus endanger the entire structure. In order to obtain the stress fields in the changed configuration, one must analyze the cylinder as an eccentric Lamé problem.

A complete solution to this problem, given by Jeffery (1921) and referenced in Timoshenko and Goodier (1970), was obtained using bipolar coordinates. However, using such a coordinate system, the solution yields stress components whose orientations are in directions which themselves are dependent on the given eccentricity. While this may not pose a severe problem, the results thus obtained lead to some difficulty in the physical interpretation of the stress fields.

If the eccentricity of the internal hole is not large, it is then possible to analyze the problem by means of perturbation techniques which lead to relatively simple solutions. Such an analysis is considered below, based on a higher-order Boundary Perturbation Method developed by Parnes and Beltzer (1986). This method, when developed to second-order, has been shown by Parnes (1987) to yield results of great accuracy even for relatively moderate eccentricities. In the following investigation we extend the method to a Method of Equivalent Perturbed Domains and apply it using a third-order scheme. The stress fields with respect to the original polar coordinate system are then calculated for various eccentricities and ratios of inner to outer radius and are presented in graphical form. The maximum stresses obtained from the method of Equivalent Perturbed Domains are compared with the results given by Jeffery and are seen to be highly accurate for moderate eccentricities.

It thus appears that the method of Equivalent Perturbed Domains as developed in this paper can lead to solutions for other problems which may prove intractable when using bipolar coordinates.

2. GENERAL FORMULATION

We consider an elastic thick wall cylinder in a domain Ω_0 bounded by a circle C_0 of radius a with center $\bar{0}$, the origin of an $(\bar{r}, \bar{\theta})$ polar coordinate system. The cylinder contains

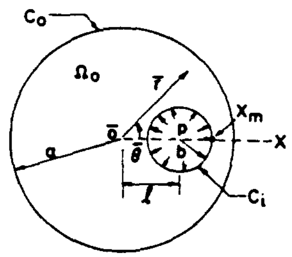

Fig. 1. Geometry of the problem.

a circular hole C_i of radius b, whose center 0 has an eccentricity with respect to $\bar{0}$ given by l. For $b < a$ and $l < a$, we define the non-dimensional parameters

$$\gamma = b/a, \quad \eta = l/a \tag{1a, b}$$

with

$$\gamma + \eta < 1. \tag{1c}$$

An internal pressure p is assumed to act along the surface C_i (Fig. 1).

Denoting the stress components in this coordinate system by $\bar{\sigma}_{ij}(\bar{r}, \bar{\theta})$, the problem is governed by a bi-harmonic equation on the Airy Stress function $\Phi(\bar{r}, \bar{\theta})$,

$$\nabla^4 \Phi(\bar{r}, \bar{\theta}) = 0 \tag{2}$$

subject to a traction-free surface C_0 and a pressure p on C_i. We consider here the case of a circular hole with small to moderate eccentricities which permits a treatment of the problem by means of a perturbation scheme.

A second coordinate system (r, θ) is first established with center at 0. If the eccentricity parameter η is sufficiently small, we may then, following the development of Parnes and Beltzer (1986), consider the boundary surface C_0 to be a perturbation of a circle C_a of radius a, and with center 0. The boundary C_a thus defines a domain Ω_a with points P_0 on C_0 being mappings of points P_a on C_a (Fig. 2). It is noted that C_0 is then a curve with varying radial distance r_0 from 0; i.e., $r_0 = r_0(\theta)$. Symbolically, the perturbed relation $C_a \to C_0$ is written as

$$a \to r_0 = r_0(a, \theta, \eta) \tag{3}$$

with $r_0|_{\eta \to 0} = a$.

We now extend the function analytically and assume that the governing equation is valid in the domain $\Omega_0 \cup \Omega_a$. Defining the non-dimensional coordinates

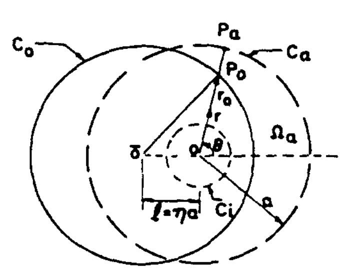

Fig. 2. Perturbation of the geometry.

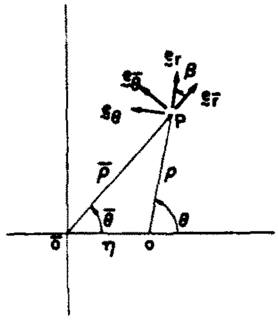

Fig. 3. Coordinate systems.

$$\bar{\rho} = \bar{r}/a, \quad \rho = r/a, \tag{4a, b}$$

the problem is then governed by the equation

$$\nabla^4 \Phi(\rho, \theta) = 0, \tag{5}$$

subject to the traction-free boundary condition on C_0

$$\overset{\bar{r}}{\mathbf{T}}(r_0, \theta) = 0, \tag{6}$$

and to the following conditions on C_i:

$$\sigma_{rr}(\gamma, \theta) = -p, \quad \sigma_{r\theta}(\gamma, \theta) = 0. \tag{7a, b}$$

Denoting unit vectors in the $(\bar{r}, \bar{\theta})$ and (r, θ) systems by $(\mathbf{e}_{\bar{r}}, \mathbf{e}_{\bar{\theta}})$ and $(\mathbf{e}_r, \mathbf{e}_\theta)$ respectively (Fig. 3),

$$\mathbf{e}_{\bar{r}} = \cos\beta\mathbf{e}_r - \sin\beta\mathbf{e}_\theta \tag{8a}$$

$$\mathbf{e}_{\bar{\theta}} = \sin\beta\mathbf{e}_r + \cos\beta\mathbf{e}_\theta \tag{8b}$$

where

$$\beta = \theta - \bar{\theta}. \tag{9}$$

Noting the stress tensor in the (r, θ) system,

$$\tau = \begin{bmatrix} \sigma_{rr}\mathbf{e}_r\mathbf{e}_r & \sigma_{r\theta}\mathbf{e}_r\mathbf{e}_\theta \\ \sigma_{\theta r}\mathbf{e}_\theta\mathbf{e}_r & \sigma_{\theta\theta}\mathbf{e}_\theta\mathbf{e}_\theta \end{bmatrix} \tag{10}$$

the traction $\overset{\bar{r}}{\mathbf{T}}$ on the surface C_0 is given by

$$\overset{\bar{r}}{\mathbf{T}} = \tau \cdot \mathbf{e}_{\bar{r}}. \tag{11}$$

Substituting eqn (8),

$$\overset{\bar{r}}{\mathbf{T}}(r_0, \theta) = \overset{\bar{r}}{T}_r\mathbf{e}_r + \overset{\bar{r}}{T}_\theta\mathbf{e}_\theta \tag{12}$$

where the components $\overset{\bar{r}}{T}_r$ and $\overset{\bar{r}}{T}_\theta$ are given by

$$\overset{f}{T}_r = \sigma_{rr}\cos\beta - \sigma_{r\theta}\sin\beta|_{C_0} \tag{13a}$$

$$\overset{f}{T}_\theta = \sigma_{r\theta}\cos\beta - \sigma_{\theta\theta}\sin\beta|_{C_0}. \tag{13b}$$

The problem is therefore specified by eqn (5) and the conditions eqns (6) and (7) where $\overset{f}{\mathbf{T}}|_{C_0}$ is defined by the components of eqns (13).

Following the perturbation scheme, the stress function $\Phi(r, \theta)$ is expanded in powers of the eccentricity η; i.e., we let

$$\Phi(r,\theta) = \Phi^{(0)} + \eta\Phi^{(1)} + \eta^2\Phi^{(2)} + \eta^3\Phi^{(3)} + \cdots = \sum_{k=0}^{N} \eta^k \Phi^{(k)}(r,\theta) \tag{14}$$

where N is the order of the scheme. The corresponding stresses $\sigma_{ij}(r, \theta)$ are assumed to be expandable as

$$\sigma_{ij}(r,\theta) = \sigma_{ij}^{(0)} + \eta\sigma_{ij}^{(1)} + \cdots = \sum_{k=0}^{N} \eta^k \sigma_{ij}^{(k)}(r,\theta) \tag{15}$$

where (Fung, 1965)

$$\sigma_{rr}^{(k)} = \frac{1}{r}\frac{\partial\Phi^{(k)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi^{(k)}}{\partial\theta^2} \tag{16a}$$

$$\sigma_{\theta\theta}^{(k)} = \frac{\partial^2\Phi^{(k)}}{\partial r^2} \tag{16b}$$

$$\sigma_{r\theta}^{(k)} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi^{(k)}}{\partial\theta}\right). \tag{16c}$$

Using the linearity property of the system, eqns (5)–(7) are satisfied by setting

$$\nabla^4\Phi^{(k)}(r,\theta) = 0, \quad k = 0, 1, 2, \ldots, N \tag{17}$$

subject to the following conditions:

For $k = 0$

$$\sigma_{rr}^{(0)}(\gamma,\theta) = -p, \quad \sigma_{r\theta}^{(0)}(\gamma,\theta) = 0 \tag{18a, b}$$

$$\overset{f}{T}_r(r_0,\theta) = \overset{f}{T}_\theta(r_0,\theta) = 0. \tag{18c, d}$$

For $k \geqslant 1$:

$$\sigma_{rr}^{(k)}(\gamma,\theta) = \sigma_{r\theta}^{(k)}(\gamma,\theta) = 0 \tag{19a, b}$$

$$\overset{f}{T}_r(r_0,\theta) = \overset{f}{T}_\theta(r_0,\theta) = 0. \tag{19c, d}$$

We observe that the boundary conditions of eqns (18c, d) and (19c, d) are specified on the contour C_0, described by the parametric relation of the form given by eqn (3). Solutions of the system of equations (17)–(19) require that the conditions of eqns (18c, d), (19c, d) be expressed along the coordinate surface C_a; i.e. we require boundary conditions for the equivalent domain Ω_a. These are obtained in the following section where, upon establishing the appropriate geometric relations and expansions, the explicit equivalent boundary conditions are derived.

3. GEOMETRIC RELATIONS AND EXPLICIT PERTURBATION EXPRESSIONS

As discussed above, we obtain solutions to the problem posed in the domain Ω_0 as a perturbed solution in the domain Ω_a. More specifically, we consider functions, and in particular the boundary conditions at a point $P_0(r = r_0, \theta)$ on C_0 as perturbed values of the functions at $P_a(r = a, \theta)$ on C_a (Fig. 2). Noting that P_0 and P_a possess the same coordinate θ, analytic functions $f(r, \theta)$ may be expanded about the point P_a by means of a Taylor series,

$$f|_{C_0} \equiv f(r_0, \theta) = f(a, \theta) + f_{,r}(a, \theta)\cdot(r_0 - a) + \tfrac{1}{2}f_{,rr}(a, \theta)\cdot(r_0 - a)^2 + \tfrac{1}{6}f_{,rrr}(a, \theta)\cdot(r_0 - a)^3 + \cdots. \quad (20)\dagger$$

We note that here $f(r_0, \theta)$ may represent either of the traction components $\overset{f}{T}_r$ or $\overset{f}{T}_\theta$, given by eqns (13), which are seen to be also dependent on $\beta = \theta - \bar{\theta}$. Before proceeding, we require explicit geometric expressions and expansions relating the two coordinate systems. In what follows below, all expressions will be expanded, unless otherwise noted, up to third order, $N = 3$.

3.1. *Geometric relations*

From Fig. 3, and the definition of eqns (4), the exact relations

$$\bar{\rho} = [\rho^2 + 2\cos\theta\rho\eta + \eta^2]^{1/2} \quad (21a)$$

$$\sin\beta = \frac{\eta}{\bar{\rho}}\sin\theta \quad (21b)$$

are noted, from which it follows that

$$\cos\beta = \frac{1}{\bar{\rho}}(\rho + \eta\cos\theta). \quad (21c)$$

For $\eta/\rho \ll 1$, expansions in powers of η/ρ up to order $N = 3$ yield

$$\bar{\rho} = \rho\left[1 + \cos\theta\eta/\rho + \frac{\sin^2\theta}{2}\eta^2/\rho^2 - \frac{1}{2}\cos\theta\sin^2\theta\eta^3/\rho^3\right] + O(\eta^4) \quad (22a)$$

as well as its reciprocal

$$\frac{1}{\bar{\rho}} = \frac{1}{\rho}\left[1 - \cos\theta\eta/\rho + \frac{1}{2}(2\cos^2\theta - \sin^2\theta)\eta^2/\rho^2 + \frac{1}{2}\cos\theta(3\sin^2\theta - 2\cos^2\theta)\eta^3/\rho^3\right] + O(\eta^4). \quad (22b)$$

Hence

$$\cos\beta = 1 - \tfrac{1}{2}\sin^2\theta\eta^2/\rho^2 + \cos\theta\sin^2\theta\eta^3/\rho^3 + O(\eta^4) \quad (23a)$$

$$\sin\beta = \sin\theta[\eta/\rho - \cos\theta\eta^2/\rho^2 + \tfrac{1}{2}(3\cos^2\theta - 1)\eta^3/\rho^3] + O(\eta^4). \quad (23b)$$

In addition, we note, upon setting $\bar{\rho} = 1(\bar{r} = a)$, that eqn (21a) leads to the exact relation

$$\rho|_{C_0} \equiv r_0/a = -\eta\cos\theta + [1 - \eta^2\sin^2\theta]^{1/2} \quad (24)$$

† Here, and in all subsequent expressions, derivatives with respect to a variable are denoted by a subscript preceded by a comma; e.g., $f_{,r} \equiv \partial f/\partial r$, etc.

whose expansion for $\eta \ll 1$ becomes

$$r_0(\theta)/a = 1-\cos\theta\eta-\frac{\sin^2\theta}{2}\eta^2+O(\eta^4) \tag{25a}$$

from which

$$\frac{1}{r_0} = \frac{1}{a}\left[1+\cos\theta\eta+\frac{1}{2}(1+\cos^2\theta)\eta^2+\cos\theta\eta^3\right]+O(\eta^4). \tag{25b}$$

Equations (25) are thus the parametric representation of the curve C_0 represented symbolically by eqn (3). It is observed here that the expressions (22) and (23) are expressed in terms of the (r, θ) system.

3.2. *Perturbation expressions*

The equivalent traction components on the boundary C_a are obtained by substituting eqns (23) in the expressions for $\overset{r}{T}_r$ and $\overset{r}{T}_\theta$ given by eqn (13), expanding according to eqn (20) and making use of eqn (25). Performing these operations and collecting in powers of η yields, for the components $\overset{r}{T}_j$, $(j = r, \theta)$:

$$\begin{aligned}\overset{r}{T}_j(r_0,\theta) = {} & \sigma_{rj}-(a\cos\theta\sigma_{rr,r}+\sin\theta\sigma_{\theta j})\eta+\frac{1}{2}[a^2\cos^2\theta\sigma_{rj,rr}-\sin^2\theta(a\sigma_{rj,r}+\sigma_{rj}) \\ & +a\sin 2\theta\sigma_{\theta j,r}]\eta^2+\frac{1}{2}\left\{a\cos\theta\left[-\frac{a^2}{3}\cos^2\theta\sigma_{rj,rrr}+\sin^2\theta(a\sigma_{rj,rr}+\sigma_{rj,r})\right]\right. \\ & \left.+\sin\theta[\cos^2\theta(3\sigma_{\theta j}-a^2\sigma_{\theta j,rr})+\sin^2\theta(a\sigma_{\theta j,r}-\sigma_{\theta j})]\right\}\eta^3+O(\eta^4).\end{aligned} \tag{26}$$

Finally, substituting the perturbation expansion, eqn (15), for the stress components σ_{ij} and again collecting in powers of η leads, for $j = r, \theta$, to the following expression:

$$\begin{aligned}\overset{r}{T}_j(r_0,\theta) = {} & \sigma_{rj}^{(0)}|_{C_a}+[\sigma_{rj}^{(1)}+{}_j\Sigma_1^{(0)}]_{C_a}\eta+[\sigma_{rj}^{(2)}+{}_j\Sigma_1^{(1)}+{}_j\Sigma_2^{(0)}]_{C_a}\eta^2 \\ & +[\sigma_{rj}^{(3)}+{}_j\Sigma_1^{(2)}+{}_j\Sigma_2^{(1)}+{}_j\Sigma_3^{(0)}]_{C_a}\eta^3\end{aligned} \tag{27}$$

where the functions ${}_j\Sigma_m^{(k)}$ $(m = 1, 2, 3; k = 0, 1, 2)$, evaluated along C_a, i.e. for $\rho = 1$, are

$${}_j\Sigma_1^{(k)} = -[a\sigma_{rj,r}\cos\theta+\sigma_{\theta j}\sin\theta]^{(k)} \tag{28a}$$

$${}_j\Sigma_2^{(k)} = \tfrac{1}{4}[a^2(1+\cos 2\theta)\sigma_{rj,rr}-(1-\cos 2\theta)(a\sigma_{rj,r}+\sigma_{rj})+2a\sin 2\theta\sigma_{\theta j,r}]^{(k)} \tag{28b}$$

$$\begin{aligned}{}_j\Sigma_3^{(k)} = {} & \tfrac{1}{24}[-a^3(3\cos\theta+\cos 3\theta)\sigma_{rj,rrr}+3a(\cos\theta-\cos 3\theta)(a\sigma_{rj,rr}+\sigma_{rj,r}) \\ & -3a^2(\sin\theta+\sin 3\theta)\sigma_{\theta j,rr}+3a(3\sin\theta-\sin 3\theta)\sigma_{\theta j,r}]^{(k)}.\end{aligned} \tag{28c}$$

The bracketed terms $[\ldots]^{(k)}$ appearing above denote that the combinations of stress components and their derivatives contained within, refer to $\sigma_{ij}^{(k)}$.

Noting in each case, $k = 0, 1, 2, 3$, that the required boundary conditions are, according to eqns (18c, d) and (19c, d), $\bar{T}_j(r_0, \theta) = 0$, these conditions are satisfied by setting the coefficients of η, appearing for all arbitrary η in eqn (27), to zero.

Thus, to summarize the scheme explicitly, the perturbation solution is obtained according to eqns (17)–(19), from solutions to the following sequence of problems:

$k = 0$:

$$\nabla^4 \Phi^{(0)} = 0 \tag{29}$$

$$\sigma_{rr}^{(0)}(\gamma, \theta) = -p, \quad \sigma_{r\theta}^{(0)}(\gamma, \theta) = 0 \tag{30a, b}$$

$$\sigma_{rr}^{(0)}(1, \theta) = \sigma_{r\theta}^{(0)}(1, \theta) = 0 \tag{30c, d}$$

$k \geqslant 1$:

$$\nabla^4 \Phi^{(k)} = 0 \tag{31}$$

$$\sigma_{rr}^{(k)}(\gamma, \theta) = \sigma_{r\theta}^{(k)}(\gamma, \theta) = 0 \tag{32a, b}$$

where for

$k = 1$:

$$\sigma_{rr}^{(1)}(1, \theta) = -{}_r\Sigma_1^{(0)}, \quad \sigma_{r\theta}^{(1)}(1, \theta) = -{}_\theta\Sigma_1^{(0)} \tag{33a, b}$$

$k = 2$:

$$\sigma_{rr}^{(2)}(1, \theta) = -[{}_r\Sigma_1^{(1)} + {}_r\Sigma_2^{(0)}]$$

$$\sigma_{r\theta}^{(2)}(1, \theta) = -[{}_\theta\Sigma_1^{(1)} + {}_\theta\Sigma_2^{(0)}] \tag{34a, b}$$

$k = 3$:

$$\sigma_{rr}^{(3)}(1, \theta) = -[{}_r\Sigma_1^{(2)} + {}_r\Sigma_2^{(1)} + {}_r\Sigma_3^{(0)}]$$

$$\sigma_{r\theta}^{(3)}(1, \theta) = -[{}_\theta\Sigma_1^{(2)} + {}_\theta\Sigma_2^{(1)} + {}_\theta\Sigma_3^{(0)}]. \tag{35a, b}$$

Each of the above cases represents an auxiliary problem whose solution yields stresses within the domain Ω_a, and by analytic extension within the domain $\Omega_0 \cup \Omega_a$. The total stress solution is then given by eqn (15).

4. PERTURBED SOLUTIONS

From the previous section, we observe that the case $k = 0$ represents a cylinder with a traction-free outer surface C_a of radius a containing a concentric hole C_i of radius b subjected to an internal pressure p (Fig. 4a). On the other hand, cases $k > 0$ describe the

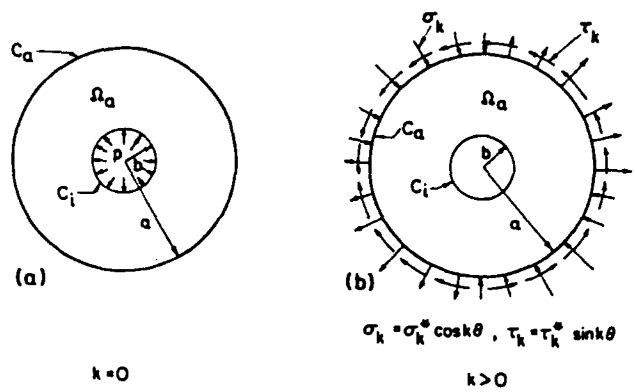

Fig. 4. Loadings of the equivalent domains, $k \geqslant 0$.

same cylindrical tube with a traction-free interior surface C_i, but subjected to tractions acting on the outer surface C_a given by eqns (33)–(35). The loadings for $k > 0$ may thus be considered as corrective tractions acting on the equivalent domain Ω_a which are required to correct the solution for the original cylinder lying in the domain Ω_0 and containing the eccentric hole. These corrective external tractions, for the cases $k > 0$ are seen, according to eqns (28), to be combinations of the form $\sigma_{rr} = \sigma_n^* \cos n\theta$, $\sigma_{r\theta} = \tau_n^* \sin n\theta$ $(n = 0, 1, 2, \ldots)$ (Fig. 4b). The general solution to these auxiliary problems is derived in the Appendix.

We proceed now to the required specific solution for each of the cases $k = 0, 1, 2, 3$.

4.1. *The* k = 0 *case*

As noted above, this case, governed by eqns (29) and (30), is recognized as the classical Lamé problem of a hollow tube subjected to an internal pressure, whose solution is (Lamé, 1852)

$$\sigma_{rr}^{(0)} = -\frac{p}{D}\left(\frac{1}{\rho^2} - 1\right) \tag{36a}$$

$$\sigma_{\theta\theta}^{(0)} = \frac{p}{D}\left(\frac{1}{\rho^2} + 1\right) \tag{36b}$$

$$\sigma_{r\theta}^{(0)} = 0 \tag{36c}$$

where

$$D = \frac{1}{\gamma^2} - 1 \tag{37}$$

with γ defined by eqn (1a).

4.2. *The* k = 1 *case*

This case is governed according to eqn (31) by the bi-harmonic equation $\nabla^4\Phi^{(1)} = 0$ with vanishing traction $\vec{T}(r, \theta) = 0$ on $r = b$. Substituting the stresses $\sigma_{ij}^{(0)}(1, \theta)$ on C_a given by eqns (36) in the remaining boundary conditions, eqn (33), the explicit loading for this case becomes

$$\sigma_{rr}^{(1)}|_{C_a} = \frac{2p}{D}\cos\theta \tag{38a}$$

$$\sigma_{r\theta}^{(1)}|_{C_a} = \frac{2p}{D}\sin\theta. \tag{38b}$$

The general solution to this external loading case is given in Section A2 of the Appendix. Hence, setting $\sigma_1^* = 2p/D$, the stresses $\sigma_{ij}^{(1)}$ are, from eqn (A16),

$$\sigma_{rr}^{(1)}(\rho, \theta) = \frac{2p}{D(1-\gamma^4)\rho^3}(\rho^4 - \gamma^4)\cos\theta \tag{39a}$$

$$\sigma_{\theta\theta}^{(1)}(\rho, \theta) = \frac{2p}{D(1-\gamma^4)\rho^3}(3\rho^4 + \gamma^4)\cos\theta \tag{39b}$$

$$\sigma_{r\theta}^{(1)}(\rho, \theta) = \frac{2p}{D(1-\gamma^4)\rho^3}(\rho^4 - \gamma^4)\sin\theta. \tag{39c}$$

4.3. *The* k = 2 *case*

This case is governed by the bi-harmonic equation on $\Phi^{(2)}$ where again the traction on the surface C_i is zero. The applied traction on C_a, obtained from the equivalent boundary conditions, eqns (34), yields, upon substitution of eqns (36) and (39), the explicit loading

$$\sigma_{rr}^{(2)}|_{C_a} = \frac{p}{D(1-\gamma^4)}[4+(3\gamma^4+1)\cos 2\theta] \tag{40a}$$

$$\sigma_{r\theta}^{(2)}|_{C_a} = \frac{p}{D(1-\gamma^4)}(5+3\gamma^4)\sin 2\theta. \tag{40b}$$

General solutions for these cases are given in the Appendix, Sections A1 and A3. Hence, letting

$$\sigma_0^* = \frac{4p}{D(1-\gamma^4)} \tag{41a}$$

and

$$\sigma_2^* = \frac{(3\gamma^4+1)p}{D(1-\gamma^4)}, \quad \tau_2^* = \frac{(5+3\gamma^4)p}{D(1-\gamma^4)} \tag{41b,c}$$

in eqns (A4) and (A19) respectively, the stresses, after simple algebraic manipulations, are readily evaluated:

$$\sigma_{rr}^{(2)} = \frac{p}{D(1-\gamma^4)(1-\gamma^2)\rho^4}\{4\rho^2(\rho^2-\gamma^2)+[(3\gamma^2+1)\rho^4-4\gamma^2\rho^2+3\gamma^4(1-\gamma^2)]\cos 2\theta\} \tag{42a}$$

$$\sigma_{\theta\theta}^{(2)} = \frac{p}{D(1-\gamma^4)(1-\gamma^2)\rho^4}\{4\rho^2(\rho^2+\gamma^2)+[(12\rho^6-(3\gamma^2+1)\rho^4-3\gamma^4(1-\gamma^2)]\cos 2\theta\} \tag{42b}$$

$$\sigma_{r\theta}^{(2)} = \frac{p}{D(1-\gamma^4)(1-\gamma^2)\rho^4}\{6\rho^6-(3\gamma^2+1)\rho^4-2\gamma^2\rho^2+3\gamma^4(1-\gamma^2)\}\sin 2\theta. \tag{42c}$$

4.4. *The* k = 3 *case*

Proceeding as in the previous cases $k = 1$ and 2, we note that this case corresponds to a cylinder, traction-free along the boundary C_i and subjected to an external loading on C_a given by eqns (35). Substituting eqns (36), (39) and (42) these become:

$$\sigma_{rr}^{(3)}|_{C_a} = \frac{2p}{D^3\gamma^4(1+\gamma^2)}\{3(1+\gamma^2)\cos\theta+(2\gamma^6-2\gamma^4+3\gamma^2-1)\cos 3\theta\} \tag{43a}$$

$$\sigma_{r\theta}^{(3)}|_{C_a} = \frac{3p}{D^3\gamma^4}\{2\sin\theta+(\gamma^4-2\gamma^2+3)\sin 3\theta\}. \tag{43b}$$

Hence, setting

$$\sigma_1^* = \tau_1^* = \frac{6p}{D^3\gamma^4} \tag{44a}$$

$$\sigma_3^* = \frac{2(2\gamma^6-2\gamma^4+3\gamma^2-1)p}{D^3\gamma^4(\gamma^2+1)}, \quad \tau_3^* = \frac{3(\gamma^4-2\gamma^2+3)p}{D^3\gamma^4} \tag{44b,c}$$

in the Appendix, and using eqns (A16) and (A18), the stresses are given by

$$\sigma_{rr}^{(3)} = \frac{\Gamma}{\rho^3}(\rho^4-\gamma^4)\cos\theta-2(3A_3\rho+2B_3\rho^3+6C_3/\rho^5+5D_3/\rho^3)\cos 3\theta \tag{45a}$$

$$\sigma_{\theta\theta}^{(3)} = \frac{\Gamma}{\rho^3}(3\rho^4+\gamma^4)\cos\theta+2(3A_3\rho+10B_3\rho^3+6C_3/\rho^5+D_3/\rho^3)\cos 3\theta \tag{45b}$$

$$\sigma_{r\theta}^{(3)} = \frac{\Gamma}{\rho^3}(\rho^4-\gamma^4)\sin\theta+6(A_3\rho+2B_3\rho^3-2C_3/\rho^5-D_3/\rho^3)\sin 3\theta \tag{45c}$$

where the constants, according to eqns (A20) and (A21), are as follows:

$$A_3 = \frac{p}{3D\kappa}(4\gamma^6+17\gamma^4+8\gamma^2+1) \tag{46a}$$

$$B_3 = -\frac{p}{D\kappa}(\gamma^4+4\gamma^2+1) \tag{46b}$$

$$C_3 = -\frac{p\gamma^6}{3D\kappa}(\gamma^6+2\gamma^4-7\gamma^2-2) \tag{46c}$$

$$D_3 = -\frac{p\gamma^4}{D\kappa}(\gamma^4+4\gamma^2+1). \tag{46d}$$

In the above,

$$\Gamma = \frac{6p}{D(1-\gamma^2)^2(1-\gamma^4)}, \quad \kappa = (\gamma^4-1)(\gamma^4+4\gamma^2+1)D^2\gamma^4. \tag{47a, b}$$

5. TOTAL SOLUTION AND TRANSFORMATIONS

We denote, for convenience, the stresses $\sigma_{ij}(r, \theta)$ and the perturbation stresses $\sigma_{ij}^{(k)}(r, \theta)$ respectively, by means of the vectors

$$\{\sigma\} = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix}, \quad \{\sigma^{(k)}\} = \begin{Bmatrix} \sigma_{rr}^{(k)} \\ \sigma_{\theta\theta}^{(k)} \\ \sigma_{r\theta}^{(k)} \end{Bmatrix}. \tag{48a, b}$$

The stress solution for the third order scheme is then, from eqn (15), written as

$$\{\sigma\} = \{\sigma^{(0)}\}+\eta\{\sigma^{(1)}\}+\eta^2\{\sigma^{(2)}\}+\eta^3\{\sigma^{(3)}\} = \sum_{k=1}^{3}\eta^k\{\sigma^{(k)}\} \tag{49}$$

where $\{\sigma^{(k)}\}$, $k = 0, 1, 2, 3$, are given respectively by eqns (36), (39), (42) and (45). It is observed that the above quantities are expressed as functions of (r, θ).

Stress components $(\sigma_{\bar{r}\bar{r}}, \sigma_{\bar{\theta}\bar{\theta}}, \sigma_{\bar{r}\bar{\theta}})$ may be obtained by means of the stress transformation laws. Denoting these components by means of the vector

$$\{\bar{\sigma}\} = \begin{Bmatrix} \sigma_{\bar{r}\bar{r}} \\ \sigma_{\bar{\theta}\bar{\theta}} \\ \sigma_{\bar{r}\bar{\theta}} \end{Bmatrix}, \tag{50}$$

the transformation law may be written as follows:

$$\{\bar{\sigma}\} = [\Lambda]\{\sigma\} \tag{51}$$

where $[\Lambda]$, the transformation matrix, is

$$[\Lambda] = \begin{bmatrix} \cos^2\beta & \sin^2\beta & -2\sin\beta\cos\beta \\ \sin^2\beta & \cos^2\beta & 2\sin\beta\cos\beta \\ \sin\beta\cos\beta & -\sin\beta\cos\beta & \cos^2\beta-\sin^2\beta \end{bmatrix} \quad (52)$$

and where the trigonometric coefficients are readily obtained from eqns (21). Expressed in this way, the stress components of $\{\bar{\sigma}\}$ with respect to the $(\bar{r}, \bar{\theta})$ system are given at field points defined by the (r, θ) coordinates. In order to express $\{\bar{\sigma}\}$ in the $(\bar{r}, \bar{\theta})$ system, it is necessary to use the inverse relations of eqns (21).

Noting that (Fig. 3)

$$\rho = [\bar{\rho}^2 - 2\cos\bar{\theta}\bar{\rho}\eta + \eta^2]^{1/2} \quad (53a)$$

and using eqn (21)

$$\sin(\theta - \bar{\theta}) = \frac{\eta}{\bar{\rho}}\sin\theta \quad (53b)$$

and the trigonometric relation

$$\theta = \tan^{-1}\left[\frac{\sin\bar{\theta}}{\cos\bar{\theta} - \eta/\bar{\rho}}\right], \quad (53c)$$

one obtains

$$\sin\theta = \frac{\bar{\rho}\sin\bar{\theta}}{\rho(\bar{\rho},\bar{\theta})}, \quad \cos\theta = \frac{\bar{\rho}\cos\bar{\theta} - \eta}{\rho(\bar{\rho},\bar{\theta})} \quad (54a, b)$$

where $\rho = \rho(\bar{\rho}, \bar{\theta})$ is given by eqn (53a). One can then readily obtain the relations

$$\sin\beta = \eta\frac{\sin\bar{\theta}}{\rho(\bar{\rho},\bar{\theta})}, \quad \cos\beta = \frac{\bar{\rho} - \eta\cos\bar{\theta}}{\rho(\bar{\rho},\bar{\theta})}. \quad (55a, b)$$

The coefficients appearing in the transformation matrix $[\Lambda]$ of eqn (52) are then readily expressed in the $(\bar{r}, \bar{\theta})$ system.

6. NUMERICAL RESULTS AND CONCLUSIONS

Stress components were calculated using the expressions of Section 4 and eqns (49)–(55). In order to verify the accuracy, numerical tests, consisting of the verification of the overall equilibrium of the upper portion of the tube ($0 \leqslant \bar{\theta} \leqslant \pi$) were performed. The tests showed the estimated accuracy of the resulting stresses to be of the order of 10% for perturbation values $\eta < 0.3$.

Results for the stress components $\sigma_{\theta\theta}$ are presented for points along the critical section, namely the x-axis (Fig. 1), for various values of γ and η.

The variation of the stresses $\sigma_{\theta\theta}$ with x for typical values of γ, $\gamma = 0.3$ and $\gamma = 0.5$, is shown in Fig. 5 for several eccentricities: $\eta = 0, 0.1, 0.15, 0.2, 0.25$.

As in the classical Lamé problem of the concentric tube, for a given hole of radius $b = \gamma a$, one observes that the maximum stress occurs on the internal boundary, C_i. For the eccentric problem at hand this maximum occurs at the point on C_i, $x_m = b + l = (\gamma + \eta)a$, which is located closest to the external boundary, that is, at the point adjacent to the smallest wall thickness.

Using eqns (36)–(47) and (15), the maximum stress at this point ($\rho = \gamma$, $\theta = 0$) is given by

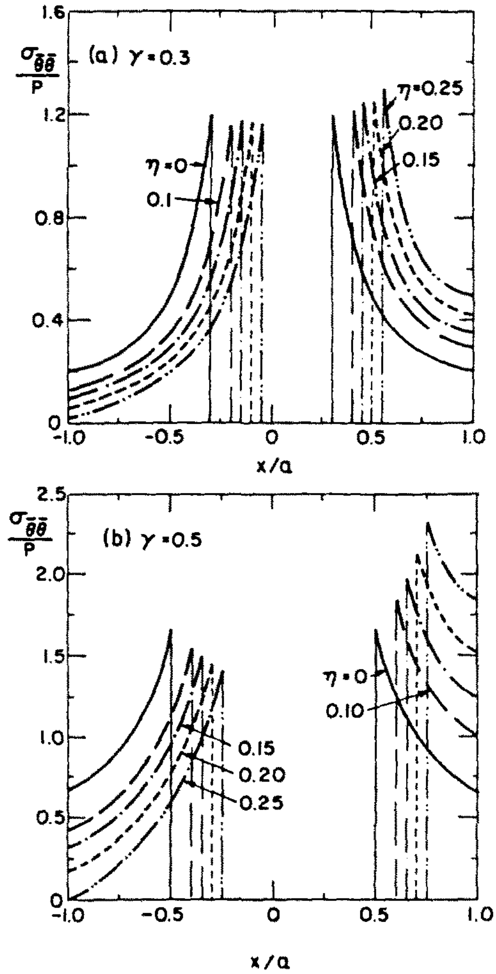

Fig. 5. Variation of the $\sigma_{\theta\theta}$ stresses along the x-axis.

$$\frac{\sigma_{\theta\theta}}{p} = \frac{1+\gamma^2}{1-\gamma^2} + \frac{8\gamma^2}{(1-\gamma^2)(1-\gamma^4)}\eta + \frac{4(1+3\gamma^3)\gamma^2}{(1-\gamma^2)^2(1+\gamma^2)}\eta^2 + \frac{16\gamma^3}{(1-\gamma^2)^4}\eta^3. \tag{56}$$

Upon expanding to third order the exact expression for the maximum stress at this point, as given by Jeffery (1921),

$$\frac{\sigma_{\theta\theta}}{p} = \frac{2(1+\gamma^2-2\gamma\eta-\eta^2)}{(1+\gamma^2)(1-\gamma^2-2\gamma\eta-\eta^2)} - 1, \qquad (57)†$$

we recover eqn (56) identically.‡

A numerical comparison of the equivalent perturbed domain results given by eqn (56), σ_{epd}, with the exact results of eqn (57), shown in Table 1, yields a measure of the accuracy of the Equivalent Perturbed Domain method as developed here. We observe that the percentage error increases for larger values of γ and, as expected, increases with greater eccentricities. Nevertheless, we may note that for moderate eccentricities, e.g. $\eta < 0.3$, the error remains within 13%. (This is consistent with the estimated order of accuracy from the overall equilibrium test as mentioned above.)

The maximum values of $\sigma_{\theta\theta}$ at the point x_m are shown in Fig. 6 as a function of γ for several eccentricities. One may note that the effect of the eccentricity η on the maximum

† The original equation as given by Jeffery (1921) is written here using the notation of the present paper.
‡ In a sense this provides a check on the expressions given in Section 4.

Table 1

	η	σ_{exa}	σ_{cpd}	% error
$\gamma = 0.1$	0.10	1.021	1.029	0.71
	0.20	1.024	1.038	1.41
	0.30	1.027	1.049	2.08
$\gamma = 0.2$	0.10	1.092	1.119	2.45
	0.20	1.106	1.159	4.79
	0.30	1.128	1.205	6.83
$\gamma = 0.3$	0.10	1.228	1.284	4.56
	0.20	1.275	1.386	8.65
	0.30	1.351	1.506	11.5
$\gamma = 0.4$	0.10	1.460	1.555	6.50
	0.20	1.586	1.771	11.7
	0.30	1.806	2.043	13.1

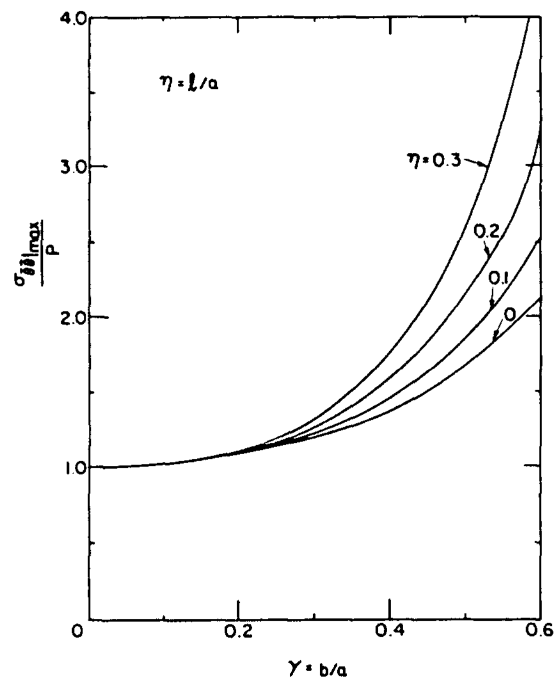

Fig. 6. Maximum $\sigma_{\theta\theta}$ stresses at X_m as a function of γ.

value increases rapidly with increased values of γ. This same effect is noted in Fig. 7 where the variation of $(\sigma_{\theta\theta})_{max}$ is presented as a function of η for a family of curves γ.

Curves showing the stress *amplification factor*, $\sigma_{\theta\theta}(x_m)/\sigma_{\theta\theta}(x_m)|_{\eta=0}$ as a function of the eccentricity η and hole size γ are presented in Fig. 8a, b. It is readily noted that the maximum stress increases by 20% in comparison to the concentric case for a hole with value $\gamma = 0.6$ which is eccentric by $l = 0.1a$. However, for this same hole, the maximum stresses increases by 200% when the eccentricity reaches $l = 0.3a$. One may conclude, using a simple physical reasoning, that for increasing values of $x_m/a = \gamma+\eta$, the stresses increase continuously and that as $x_m/a \to 1$, $\sigma_{\theta\theta} \to \infty$.

It is evident that the perturbation method used in this investigation cannot be expected to lead to accurate results for large eccentricities. Nevertheless, within the limited range of moderate eccentricities, the solutions obtained by the Method of Equivalent Domains yield reasonably accurate quantitative results for the effect of eccentricities on the stresses for various cylindrical geometries. It may be expected that the application of the method to other problems will yield solutions of similar accuracy.

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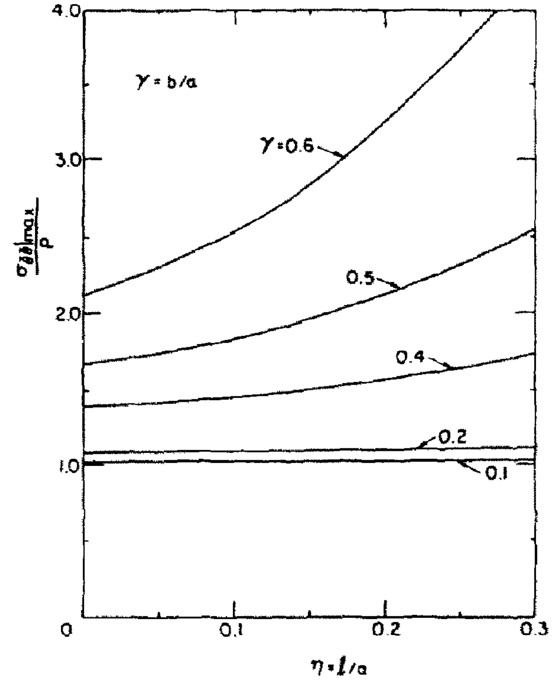

Fig. 7. Maximum $\sigma_{\theta\theta}$ stresses at X_m as a function of η.

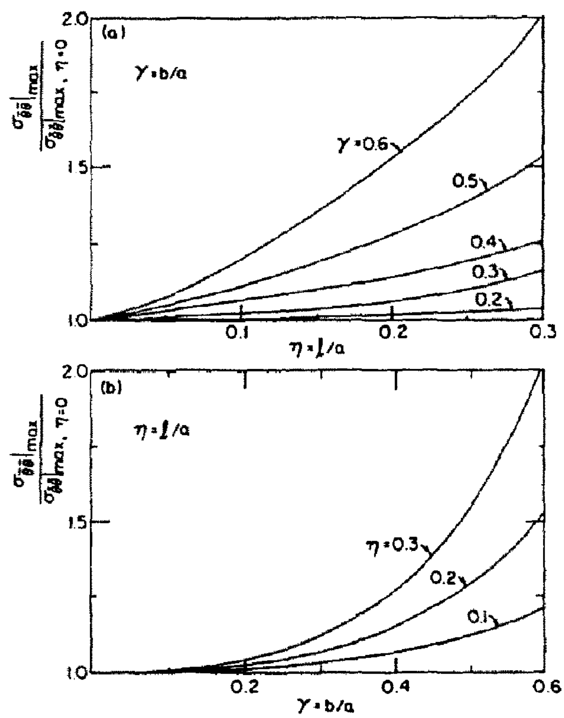

Fig. 8. Stress amplification factors.

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APPENDIX: AUXILIARY PROBLEM SOLUTIONS—HOLLOW CYLINDER SUBJECTED TO θ-DEPENDENT EXTERNAL TRACTIONS

We obtain the solutions to the required auxiliary problems: concentric hollow cylinders (with inner and outer radii b and a respectively; $b = \gamma a$; $\gamma < 1$), subjected to external cosinusoidal and sinusoidal loadings on the outer surface C_a (Fig. 4b).

The stresses can be obtained from the Airy stress function $\Phi^{(n)}(r, \theta)$ satisfying the bi-harmonic equation

$$\nabla^4\Phi^{(n)}(r,\theta), \quad n > 0 \tag{A1}$$

and subject to the conditions

$$\sigma_{rr}|_{C_i} = \sigma_{r\theta}|_{C_i} = 0, \tag{A2a, b}$$

$$\sigma_{rr}|_{C_a} = \sigma_n^*\cos n\theta, \quad \sigma_{r\theta}|_{C_a} = \tau_n^*\sin n\theta, \quad n \geqslant 0 \tag{A3a, b}$$

where σ_n^* and τ_n^* are known quantities. The resulting stress components are then given by eqns (16).

A1. *Case* n = 0

This case represents the classical Lamé problem of a hollow concentric cylinder subjected to an axisymmetric external loading $\sigma_{rr}|_{C_a} = \sigma_0^*$, whose solution is (Lamé, 1852)

$$\sigma_{rr}^{(0)} = \frac{\sigma_0^*}{D\gamma^2\rho^2}(\rho^2-\gamma^2) \tag{A4a}$$

$$\sigma_{\theta\theta}^{(0)} = \frac{\sigma_0^*}{D\gamma^2\rho^2}(\rho^2+\gamma^2) \tag{A4b}$$

$$\sigma_{r\theta}^{(0)} = 0 \tag{A4c}$$

where D is defined by eqn (37), and where $\rho = r/a$.

The cases $n = 1$ and $n > 1$ require a separate treatment.

A2. *Case* n = 1

It is first observed that overall equilibrium considerations of the cylinder lead to the requirement $\sigma_1^* = \tau_1^*$ in eqns (A3). Consistent with the boundary conditions of eqns (A2) and (A3), $\Phi^{(1)}(r, \theta)$ admits a solution of the form

$$\Phi^{(1)}(r,\theta) = \frac{\tilde{A}}{2}r\theta\sin\theta+(a_1r+A_1/r+B_1r^3+b_1r\log r)\cos\theta. \tag{A5}$$

Substituting in eqns (16)

$$\sigma_{rr}^{(1)} = \frac{\tilde{A}}{r}\cos\theta+(-2A_1/r^3+2B_1r+b_1/r)\cos\theta \tag{A6a}$$

$$\sigma_{\theta\theta}^{(1)} = (2A_1/r^3+6B_1r+b_1/r)\cos\theta \tag{A6b}$$

$$\sigma_{r\theta}^{(1)} = (-2A_1/r^3+2B_1r+b_1/r)\sin\theta. \tag{A6c}$$

Equations (A2a, b) then lead respectively to

$$\tilde{A}_1-\frac{2}{b^2}A_1+2b^2B_1+b_1 = 0 \tag{A7a}$$

$$-\frac{2}{b^2}A_1+2b^2B_1+b_1 = 0. \tag{A7b}$$

Hence $\tilde{A}_1 = 0$. Noting too that the a_1 term does not contribute to the stresses, we set $a_1 = 0$.

Substitution of the boundary conditions, eqns (A3a, b), with the above-mentioned requirement ($\sigma_1^* = \tau_1^*$), yields the identical equation

$$-2A_1/a^3+2B_1a+\frac{b_1}{a} = \sigma_1^*. \tag{A8}$$

It is noted that eqns (A7b) and (A8) contain three unknown constants. However, it is now shown that the b_1 term leads to multivalued displacements. Substituting in the stress–strain relations, e.g., in the case of plane stress, the b_1 term yields

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr}-\nu\sigma_{\theta\theta}) = \frac{b_1}{Er}(1-\nu)\cos\theta \tag{A9a}$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta}-\nu\sigma_{rr}) = \frac{b_1}{Er}(1-\nu)\cos\theta \tag{A9b}$$

$$\varepsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta} = \frac{b_1}{Er}(1+\nu)\sin\theta \tag{A9c}$$

where E is the modulus and ν Poisson's ratio of the material. Hence

$$u_r = \int \varepsilon_{rr}(r,\theta)\,\mathrm{d}r = \frac{b_1}{E}(1-\nu)\log r\cos\theta + f(\theta). \tag{A10a}$$

From the relation

$$\frac{\partial u_\theta}{\partial \theta} = r\varepsilon_{\theta\theta} - u_r = \frac{b_1}{E}(1-\nu)(1-\log r)\cos\theta - f(\theta), \tag{A10b}$$

it follows that

$$u_\theta = \frac{b_1}{E}(1-\nu)(1-\log r)\sin\theta - \int f(\theta)\,\mathrm{d}\theta + g(r). \tag{A10c}$$

Now, since

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \tag{A11}$$

we obtain, using eqns (A9c), (A10a) and (A10c), the resulting relation

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} + \int f(\theta)\,\mathrm{d}\theta + r\frac{\mathrm{d}g(r)}{\mathrm{d}r} - g(r) = \frac{4b_1}{E}\sin\theta, \tag{A12}$$

which can be satisfied for all r and θ, only if

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} + \int f(\theta)\,\mathrm{d}\theta = \frac{4b_1}{E}\sin\theta \tag{A13a}$$

and

$$r\frac{\mathrm{d}g(r)}{\mathrm{d}r} - g(r) = 0. \tag{A13b}$$

The solution to eqn (A13a) being

$$f(\theta) = \bar{A}\sin\theta + \bar{B}\cos\theta + \frac{2b_1}{E}\theta\sin\theta, \tag{A14}$$

we observe, from eqns (A.10a, c), that values $b_1 \neq 0$ necessarily lead to multi-valued displacements. We therefore set $b_1 = 0$.

The remaining constants A_1 and B_1 are then readily obtained from eqns (A7a) and (A8):

$$A_1 = \frac{\sigma_1^* \gamma^2 a^3}{2D(1+\gamma^2)}, \quad B_1 = \frac{\sigma_1^*}{2D(1+\gamma^2)a} \tag{A15a, b}$$

where use has now been made of eqn (37).

Substitution in eqns (A6) then yields

$$\sigma_{rr}^{(1)}(\rho,\theta) = \frac{\sigma_1^*}{D\gamma^2(1+\gamma^2)\rho^3}(\rho^4 - \gamma^4)\cos\theta \tag{A16a}$$

$$\sigma_{\theta\theta}^{(1)}(\rho,\theta) = \frac{\sigma_1^*}{D\gamma^2(1+\gamma^2)\rho^3}(3\rho^4 + \gamma^4)\cos\theta \tag{A16b}$$

$$\sigma_{r\theta}^{(1)}(\rho,\theta) = \frac{\sigma_1^*}{D\gamma^2(1+\gamma^2)\rho^3}(\rho^4 - \gamma^4)\sin\theta. \tag{A16c}$$

A3. *Cases* n ≥ 2

The solution $\Phi^{(n)}$ of eqn (A1) may be taken in the form

$$\Phi^{(n)}(r,\theta) = (A_n r^n + B_n r^{n+2} + C_n/r^n + D_n/r^{n-2})\cos n\theta. \tag{A17}$$

Substituting in eqns (17),

$$\sigma_{rr}^{(n)} = [-n(n-1)A_n r^{n-2} - (n^2-n-2)B_n r^n - n(n+1)C_n r^{-n-2} - (n^2+n-2)D_n r^{-n}]\cos n\theta \tag{A18a}$$

$$\sigma_{\theta\theta}^{(n)} = [n(n-1)A_n r^{n-2} + (n+2)(n+1)B_n r^n + n(n+1)C_n r^{-n-2} + (n-1)(n-2)D_n r^{-n}]\cos n\theta \tag{A18b}$$

$$\sigma_{r\theta}^{(n)} = n[(n-1)A_n r^{n-2} + (n+1)B_n r^n - (n+1)C_n r^{-n-2} - (n-1)D_n r^{-n}]\sin n\theta. \tag{A18c}$$

For $n = 2$, substitution in the boundary conditions, eqns (A2) and (A3), leads to evaluation of the four constants; omitting the algebraic details, these become:

$$A_2 = \frac{1}{2(\gamma^2-1)^3}[(2\gamma^4+\gamma^2+1)\sigma_2^* - 2\gamma^4\tau_2^*] \tag{A19a}$$

$$B_2 = \frac{1}{6(\gamma^2-1)^3a^2}[-(3\gamma^2+1)\sigma_2^* + (3\gamma^2-1)\tau_2^*] \tag{A19b}$$

$$C_2 = \frac{\gamma^4a^4}{6(\gamma^2-1)^3}[(\gamma^2+3)\sigma_2^* - 2\gamma^2\tau_2^*] \tag{A19c}$$

$$D_2 = -\frac{\gamma^2a^2}{2(\gamma^2-1)^3}[(\gamma^4+\gamma^2+2)\sigma_2^* - \gamma^2(\gamma^2+1)\tau_2^*]. \tag{A19d}$$

For the case $n = 3$, the following constants are similarly obtained:

$$A_3 = \frac{1}{12Qa}[3(3\gamma^6+\gamma^4+\gamma^2+1)\sigma_3^* + (-9\gamma^6+\gamma^4+\gamma^2+1)\tau_3^*] \tag{A20a}$$

$$B_3 = -\frac{1}{8Qa^3}[(4\gamma^4+\gamma^2+1)\sigma_3^* - (4\gamma^4-\gamma^2-1)\tau_3^*] \tag{A20b}$$

$$C_3 = -\frac{\gamma^6a^5}{24Q}[-3(\gamma^4+\gamma^2+4)\sigma_3^* + (5\gamma^4+5\gamma^2-4)\tau_3^*] \tag{A20c}$$

$$D_3 = -\frac{\gamma^4a^3}{4Q}\{[2+(\gamma^2+1)(\gamma^4+1)]\sigma_3^* + [2-(\gamma^2+1)(\gamma^4+1)]\tau_3^*\} \tag{A20d}$$

where

$$Q = (\gamma^2-1)^3(\gamma^4+4\gamma^2+1). \tag{A21}$$